CORRECTNESS OF AN ANALYTICAL SOLUTION OF AN INVERSE PROBLEM OF HEAT CONDUCTION

N. M. Tsirel'man

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We discuss the analytical solution of an inverse Cauchy boundary-value problem of heat conduction and analyze its correctness.

The analytical solution of inverse boundary-value problems, together with solution by other methods is of interest in thermal engineering [1]. In [2-4] analytical solutions were constructed in the form of functional series. An important drawback of this form is that they contain derivatives of arbitrary order of the experimentally determined time dependence of the temperature and its gradient at points inside the body. This immediately limits the applicability of series constructed in this way because errors in the measurements of the parameters are amplified when their time derivatives are calculated.

The purpose of the present paper is to demonstrate the possibility of constructing an analytical solution of inverse problems free from this difficulty and to analyze the correctness of the problem within the class of analytic functions.

1. We consider an inverse boundary-value problem in the Cauchy formulation for an infinite plate:

$$v_t = v_{xx}, \quad t > 0, \tag{1}$$

$$v(t,\ 0) = f(t), \tag{2}$$

$$v_x(t,\ 0) = \varphi(t). \tag{3}$$

We apply the integral Laplace transform with respect to x to Eqs. (1)-(3):

$$v(x,\ t) \doteqdot V(p,\ t) = \int_0^\infty v(x,\ t)\exp(-px)\,dx. \tag{4}$$

Then for the left- and right-hand sides of (1) we have

$$v_t(x,\ t) \doteqdot V_t(p,\ t), \tag{5}$$

$$v_{xx}(x,\ t) \doteqdot p^2V(p,\ t) - pf(t) - \varphi(t). \tag{6}$$

We obtain the following equation for the transform

$$V_t(p,\ t) = p^2V(p,\ t) - pf(t) - \varphi(t), \tag{7}$$

whose solution is

$$V(p,\ t) = -\exp(p^2t)\left\{\int_0^t [pf(\tilde{t}) + \varphi(\tilde{t})]\exp(-p^2\tilde{t})\,d\tilde{t} + C(p)\right\}. \tag{8}$$

Among the solutions of (7), only one of them is the correct transform. It can be obtained from the general solution (8) if we $C(p) \equiv 0$ in it (within the class of analytic functions, the function $C(p)\exp(p^2t)$ is a Laplace transform only when $C(p) \equiv 0$).

In order to find the inverse transform of the solution (8), we consider two possible cases of practical interest in which the experimental time dependence of f(t) and $\varphi(t)$ is approximated.

2. We approximate f(t) and $\varphi(t)$ by exponential series of the form

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$$f(t)=\sum_{k=0}^{n} a_k \exp(\alpha_k^2 t), \quad \varphi(t)=\sum_{k=0}^{m} b_k \exp(\beta_k^2 t) \tag{9}$$

and we then obtain for the transform

$$V(p,\ t)=\sum_{k=0}^{n} a_k \frac{p}{p^2-\alpha_k^2}\exp(\alpha_k^2 t)+\sum_{k=0}^{m} b_k \frac{\exp(\beta_k^2 t)}{p^2-\beta_k^2}. \tag{8'}$$

Then after the inverse transform, we find for the solution of (1)-(3):

$$v(x,\ t)=\sum_{k=0}^{n} a_k \exp(\alpha_k^2 t)\operatorname{ch}\alpha_k x+\sum_{k=0}^{m} b_k \frac{\exp(\beta_k^2 t)}{\beta_k}\operatorname{sh}\beta_k x. \tag{10}$$

Although this result is of considerable practical value, it is not suitable for analyzing the correctness of the solution.

3. If we approximate f(t) and $\varphi(t)$ by polynomials

$$f(t)=\sum_{k=0}^{n} a_k t^k \text{ and } \varphi(t)=\sum_{k=0}^{m} b_k t^k \tag{11}$$

we obtain for the transform

$$V(p,\ t)=-p\exp(p^2 t)\sum_{k=0}^{n} a_k \int t^k \exp(-p^2 t)\,dt-\exp(p^2 t)\sum_{k=0}^{m} b_k \int t^k \exp(-p^2 t)\,dt=$$

$$=\sum_{k=0}^{n} a_k\Big[\sum_{i=1}^{k+1}\frac{k!\,t^{k-i+1}}{(k-i+1)!\,p^{2i-1}}\Big]+\sum_{k=0}^{m} b_k\Big[\sum_{i=1}^{k+1}\frac{k!\,t^{k-i+1}}{(k-i+1)!\,p^{2i}}\Big]. \tag{12}$$

Then the solution of (1)-(3) is

$$v(x,\ t)=\sum_{k=0}^{n} a_k\Big[\sum_{i=1}^{k+1}\frac{k!\,t^{k-i+1}x^{2i-2}}{(k-i+1)!\,(2i-2)!}\Big]+\sum_{k=0}^{m} b_k\Big[\sum_{i=1}^{k+1}\frac{k!\,t^{k-i+1}x^{2i-1}}{(k-i+1)!\,(2i-1)!}\Big]. \tag{13}$$

4. We consider the correctness of the solution (13) for $0 < x < b$ and $0 < t < T$ within the class of polynomials, i.e., we put

$$f_1-f_2=\varepsilon\sum_{k=0}^{N}\tilde{a}_k t^k, \quad \varepsilon>0, \tag{14}$$

$$\varphi_1-\varphi_2=\delta\sum_{k=0}^{M}\tilde{b}_k t^k, \quad \delta>0. \tag{15}$$

We have the following upper bounds for (14) and (15):

$$\begin{aligned}|f_1-f_2|&<\varepsilon\max|\tilde{a}_k|(T^{N+1}-1)/(T-1)=\varepsilon_1(T^{N+1}-1)/(T-1),\\ |\varphi_1-\varphi_2|&<\delta\max|\tilde{b}_k|(T^{M+1}-1)/(T-1)=\delta_1(T^{M+1}-1)/(T-1).\end{aligned} \tag{16}$$

The difference between the solution (13) of the problem (1)-(3) with f_1, φ_1 and with f_2, φ_2

$$v_1-v_2=\varepsilon\sum_{k=0}^{N}\tilde{a}_k\Big[\sum_{i=1}^{k+1}\frac{k!\,t^{k-i+1}x^{2i-2}}{(k-i+1)!\,(2i-2)!}\Big]+\delta\sum_{k=0}^{M}\tilde{b}_k\Big[\sum_{i=1}^{k+1}\frac{k!\,t^{k-i+1}x^{2i-1}}{(k-i+1)!\,(2i-1)!}\Big]. \tag{17}$$

The upper bound (17) gives

$$|v_1 - v_2| < \varepsilon_1 \sum_{k=0}^{N} \Big[\sum_{i=1}^{k+1} \frac{k!\,T^{k-i+1}b^{2i-2}}{(k-i+1)!\,(2i-2)!}\Big] + \delta_1 \sum_{k=0}^{M} \Big[\sum_{i=1}^{k+1} \frac{k!\,T^{k-i+1}b^{2i-1}}{(k-i+1)!\,(2i-1)!}\Big] <$$

$$< \varepsilon_1 \sum_{k=0}^{N} F_1(T,\ b)\Big[\sum_{i=1}^{k+1} \frac{k!}{(k-i+1)!\,(2i-2)!}\Big] + \delta_1 \sum_{k=0}^{M} F_2(T,\ b)\Big[\sum_{i=1}^{k+1} \frac{k!}{(k-i+1)!\,(2i-1)!}\Big]. \tag{18}$$

Using the fact that the sums $\sum_{i=1}^{k+1} \frac{k!}{(k-i+1)!\,(2i-2)!}$ and $\sum_{i=1}^{k+1} \frac{k!}{(k-i+1)!\,(2i-1)!}$ are dominated by 2^k, we have the upper bound

$$|v_1 - v_2| < \varepsilon_1 \sum_{k=0}^{N} F_1(T,\ b)\,2^k + \delta_1 \sum_{k=0}^{M} F_2(T,\ b)\,2^k. \tag{19}$$

The functions $F_1(T,\ b)$ and $F_2(T,\ b)$ are equal to $b^{2k}T^k$ and $b^{2k+1}T^k$, respectively for $b > 1$, $T > 1$; unity for $b < 1$, $T < 1$; b^{2k}, b^{2k+1}, respectively for $b > 1$, $T < 1$; and T^k for $b < 1$, $T > 1$. We then obtain for these four cases

$$\begin{aligned}
&\text{a)}\ |v_1 - v_2| < \varepsilon_1 [(2Tb^2)^{N+1} - 1]/(2Tb^2 - 1) + \delta_1 b\,[(2Tb^2)^{M+1} - 1]/(2Tb^2 - 1),\\
&\text{b)}\ |v_1 - v_2| < \varepsilon_1 (2^{N+1} - 1) + \delta_1 (2^{M+1} - 1),\\
&\text{c)}\ |v_1 - v_2| < \varepsilon_1 [(2b^2)^{N+1} - 1]/(2b^2 - 1) + \delta_1 b\,[(2b^2)^{M+1} - 1]/(2b^2 - 1),\\
&\text{d)}\ |v_1 - v_2| < \varepsilon_1 [(2T)^{N+1} - 1]/(2T - 1) + \delta_1 [(2T)^{M+1} - 1]/(2T - 1).
\end{aligned} \tag{19'}$$

Comparison of (19') with (16) shows that the solution of the problem depends not only on the accuracy of the input data, but also on the dimensions of the space–time region.

NOTATION

$v(x,\ t)$, temperature at a point on the plate with coordinate x, $t = a\tau/l_0^2$ and τ, dimensionless time and actual time; a, thermal diffusivity of the body; l_0, characteristic length in units of which x is measured; $f(t)$ and $\varphi(t)$, time dependence of the temperature and temperature gradient measured at the point $x = 0$.

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